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Mixed convection around a horizontal circular cylinder immersed in a Darcy flow

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Abstract. The forced and free mixed convection problem around a horizontal circular cylinder in a fluid-saturated porous medium is investigated at small Peclet number with $\mathrm{Gr}/\mathrm{Re} = O(1)$. The Darcy flow model is used for the velocity field. The method of matched asymptotic expansions is used to obtain asymptotic solutions for small Peclet number. It is shown that, up to the present order of approximation, natural convection has no effects on the temperature field, and the effects of the parameter Gr/Re on the velocity field are examined in detail.

1. Introduction

The study of convection flows around heated bodies embedded in saturated porous media has been made by many authors because of their importance in geophysical and engineering applications. The present paper concerns with the forced and free mixed convection flow around a horizontal circular cylinder immersed in a porous medium through which a liquid is flowing according to Darcy's law. Most of the previous works concerned with the mixed convection around a body considered the case of wedge configuration including a flat plate as a special case. The work by Cheng [1] is the only work which is related to the mixed convection around a horizontal circular cylinder, as far as the present author knows. He used a boundary layer formulation and showed that with a generalized similarity transformation, the resulting ordinary differential equation and boundary conditions reduce to those of a vertical isothermal surface, for which similarity solutions have already been obtained [2]. In the present paper, we consider the mixed convection around a horizontal circular cylinder in a porous medium at small Peclet number with $\mathrm{Gr}/\mathrm{Re} = O(1)$, where Gr and Re are the Grashoff and Reynolds numbers, respectively. As Bejan [3] stated, the study of the low Peclet number case is more appropriate in view of the limitations associated with using the Darcy flow model. The solutions are obtained using the method of matched asymptotic expansions, in which the velocity and temperature are, respectively, expressed as separate, locally valid, expansions in terms of Pe (Peclet number) for two regions, namely, the inner and outer regions [4].

2. Governing equations

Consider a Darcy flow around a heated (or cooled) horizontal circular cylinder of radius a. The superficial velocity (volumetric flow per unit cross-section area) of the far upstream is assumed to be uniform ($=U_\infty$) and in the opposite direction of gravity. The non-dimensional equations governing the mixed convection flow around a cylinder can be written under the

Boussinesq approximation as

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}=0\,, \tag{1}$$

$$u=-\frac{\partial p}{\partial r}-\frac{\mathrm{Gr}}{\mathrm{Re}}\,t\cos\theta\,, \tag{2a}$$

$$v=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{\mathrm{Gr}}{\mathrm{Re}}\,t\sin\theta\,, \tag{2b}$$

$$u\frac{\partial t}{\partial r}+\frac{v}{r}\frac{\partial t}{\partial \theta}=\mathrm{Pe}^{-1}\,\nabla^2 t\,, \tag{3}$$

where

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

In the above equations, ($r(=r'/a)$, θ) is cylindrical coordinates with $r=0$ at the center of the cylinder and $\theta=0$ in the direction of gravity, $u(=u'/U_\infty)$ and $v(=v'/U_\infty)$ the superficial velocities in r- and θ-direction, respectively, $p(=K(p'-\rho_\infty gz)/a\mu U_\infty)$ the pressure, $t(=(t'-T_\infty)/(T_w-T_\infty))$ the temperature,

$$\mathrm{Gr}=\frac{Kg\beta a(T_w-T_\infty)}{\nu^2}\,, \tag{4a}$$

$$\mathrm{Re}=\frac{U_\infty a}{\nu}\quad\text{and}\quad \mathrm{Pe}=\frac{\rho_\infty C_p U_\infty a}{\lambda_e}\,, \tag{4b}$$

where prime denotes the dimensional quantities, K is the medium permeability, ρ_∞ the density of the fluid at infinity, c_p the specific heat of the fluid, g the gravity constant, μ the viscosity of the fluid, β the volumetric coefficient of expansion, T_w the surface temperature of the cylinder, λ_e the effective thermal conductivity of the saturated porous medium and $\nu=\mu/\rho_\infty$.

Introducing the stream function ψ defined by

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\,,\quad v=\frac{\partial\psi}{\partial r}\,, \tag{5}$$

the equation of continuity is automatically satisfied and eq. (2) may be written as

$$\nabla^2\psi=\frac{\mathrm{Gr}}{\mathrm{Re}}\left(\frac{1}{r}\frac{\partial t}{\partial\theta}\cos\theta+\frac{\partial t}{\partial r}\sin\theta\right)\,. \tag{6}$$

The boundary conditions are

$$\psi=0\,,\quad t=1\quad \text{at}\quad r=1\,, \tag{7a}$$

$$\psi\to r\sin\theta\,,\quad t\to 0\quad\text{as}\quad r\to\infty\,. \tag{7b}$$

3. Asymptotic solutions for small Peclet number

We shall now proceed to obtain asymptotic solutions of (3) and (6) for small Peclet numbers. We assume that t and ψ may be expanded as

$$t = t_0 + \Delta t_1 + \Delta^2 t_2 + \cdots + O(\mathrm{Pe}) , \tag{8}$$

$$\psi = \psi_0 + \Delta\psi_1 + \Delta^2\psi_2 + \cdots + O(\mathrm{Pe}) , \tag{9}$$

respectively, where

$$\Delta = 1/(\ln(4/\mathrm{Pe}) - C) , \tag{10}$$

$C = 0.5772157\ldots$ being Euler's constant. These assumptions are similar to those for other two-dimensional flow problems at low Peclet numbers. Substitution of (8) and (9) into (3) and (6) yields

$$\nabla^2 t_n = 0 \tag{11}$$

and

$$\nabla^2\psi_n = \frac{\mathrm{Gr}}{\mathrm{Re}}\left(\frac{1}{r}\frac{\partial t_n}{\partial\theta}\cos\theta + \frac{\partial t_n}{\partial r}\sin\theta\right) . \tag{12}$$

Equation (11) is the heat conduction equation and its solution satisfying the boundary condition on the surface is

$$t_0 = a_0 \ln r + 1 , \tag{13a}$$

$$t_n = a_n \ln r \text{ (for } n \geqq 1) , \tag{13b}$$

where a_n are integral constants. Substituting (13) into (12), we have

$$\nabla^2\psi_n = a_n \frac{\mathrm{Gr}}{\mathrm{Re}} \frac{\sin\theta}{r} . \tag{14}$$

The solution of this equation satisfying the boundary condition on the surface is

$$\psi_n = \frac{a_n}{2}\frac{\mathrm{Gr}}{\mathrm{Re}} r \ln r \sin\theta + \sum_{k=1}^{\infty} b_{nk}(r^k - r^{-k}) \sin k\theta , \tag{15}$$

where b_{nk} are integral constants. It is to be noted here that, in view of the solution (13), the expansion for t, eq. (8), could not be made to vanish at infinity, suggesting that the expansion (8) and therefore expansion (9) are valid only in the inner region where $r = O(1)$. This failure of the expansion (8) for large r is closely similar to what is generally observed in the problems of low Reynolds (or Peclet) number flow around a body, and can be explained by the fact that, in the outer region far from the body where $r = O(\mathrm{Pe}^{-1})$, the convection terms in the energy equation become comparable order of magnitude with the conduction terms.

In the outer region, we introduce the following variables

$$\rho = \text{Pe}\, r\,, \tag{16a}$$

$$\Psi(\rho, \theta) = \text{Pe}\, \psi(r, \theta),\ T(\rho, \theta) = t(r, \theta)/\delta(\text{Pe})\,, \tag{16b}$$

where $\delta(\text{Pe})$ is the still unknown order of the temperature in the outer region. In terms of these outer variables, equations for Ψ and T may be written as

$$\nabla_\rho^2 \Psi = \delta(\text{Pe}) \frac{\text{Gr}}{\text{Re}} \left(\frac{1}{\rho} \frac{\partial T}{\partial \theta} \cos\theta + \frac{\partial T}{\partial \rho} \sin\theta \right), \tag{17}$$

$$U \frac{\partial T}{\partial \rho} + \frac{V}{\rho} \frac{\partial T}{\partial \theta} = \nabla_\rho^2 T\,, \tag{18}$$

where

$$U = -\rho^{-1}\, \partial\Psi/\partial\theta\,, \quad V = \partial\Psi/\partial\rho\,, \tag{19}$$

and ∇_ρ^2 is the same operator as ∇^2, but with r replaced by ρ. The solutions for (17) and (18) are assumed to be of the following forms,

$$\Psi = \Psi_0 + \Delta\Psi_1 + \Delta^2\Psi_2 + \cdots + O(\text{Pe})\,, \tag{20}$$

$$T = T_0 + \Delta T_1 + \Delta^2 T_2 + \cdots + O(\text{Pe})\,. \tag{21}$$

These outer expansions are required to satisfy the boundary conditions at infinity and, instead of satisfying the boundary conditions on the surface, to satisfy the following matching conditions

$$\lim_{\rho\to 0} \Psi(\rho, \theta) = \text{Pe} \lim_{r\to\infty} \psi(r, \theta)\,, \tag{22a}$$

$$\lim_{\rho\to 0} T(\rho, \theta) = \{1/\delta(\text{Pe})\} \lim_{r\to\infty} t(r, \theta)\,. \tag{22b}$$

From these matching conditions together with the requirement that both T and Ψ should be of order unity, we can easily show that

$$\delta(\text{Pe}) = \Delta\,, \tag{23a}$$

$$a_0 = 0\,, \quad a_1 = -1\,, \quad b_{nk} = 0 \quad \text{for} \quad k \geqq 2\,, \tag{23b}$$

and that, Ψ and T should behave, as $\rho \to 0$, as

$$\begin{aligned} \Psi \sim & \left(b_{01} - \frac{1}{2}\frac{\text{Gr}}{\text{Re}}\right) \rho \sin\theta + \cdots \\ & + \Delta\left\{ -\frac{1}{2}\frac{\text{Gr}}{\text{Re}} \rho \ln \rho \sin\theta + \left(b_{11} - \frac{1}{2}\frac{\text{Gr}}{\text{Re}}[C - \ln 4]\right) \rho \sin\theta + \cdots \right\} + O(\Delta^2)\,, \end{aligned} \tag{24}$$

$$T \sim -\ln \rho + (a_2 + \ln 4 - C) + \cdots$$
$$+ \Delta\{a_2 \ln \rho + a_3 + a_2(C - \ln 4) + \cdots\} + O(\Delta^2) . \tag{25}$$

The fact that $b_{nk} = 0$ for $k \geqq 2$ suggests that the inner streamlines are symmetrical with respect to the midplane $\theta = \pi/2$ and $3\pi/2$. The asymmetry of the streamlines will appear only when the terms of $O(\mathrm{Pe})$ are included in the analysis.

Inserting (19) and (20) into (17) and (18), we can obtain the equations for Ψ_0, Ψ_1 and T_0 as

$$\nabla_\rho^2 \Psi_0 = 0 , \tag{26}$$

$$\nabla_\rho^2 \Psi_1 = \frac{\mathrm{Gr}}{\mathrm{Re}} \left(\frac{1}{\rho} \frac{\partial T_0}{\partial \theta} \cos \theta + \frac{\partial T_0}{\partial \rho} \sin \theta \right) , \tag{27}$$

$$-\frac{1}{\rho} \frac{\partial \Psi_0}{\partial \theta} \frac{\partial T_0}{\partial \rho} + \frac{1}{\rho} \frac{\partial \Psi_0}{\partial \rho} \frac{\partial T_0}{\partial \theta} = \nabla_\rho^2 T_0 . \tag{28}$$

The solution of (26) satisfying the boundary condition at infinity is apparently a uniform stream

$$\Psi_0 = \rho \sin \theta , \tag{29}$$

and the equation for T_0 becomes

$$-\cos \theta \frac{\partial T_0}{\partial \rho} + \frac{\sin \theta}{\rho} \frac{\partial T_0}{\partial \theta} = \nabla_\rho^2 T_0 . \tag{30}$$

The general solution of (30) satisfying the boundary condition at infinity is

$$T_0 = A_0 K_0\left(\frac{\rho}{2}\right) \exp(\rho\mu/2) , \quad \mu = -\cos \theta , \tag{31}$$

where A_0 is an integral constant and $K_0(z)$ the modified Bessel function of the second kind. The constant A_0 as well as the constant a_2 can be determined from the matching condition (25) as

$$A_0 = 1 , \quad a_2 = 0 . \tag{32}$$

Thus, T_0, t_0, t_1 and t_2 have been determined completely as

$$T_0 = K_0(\rho/2) \exp(\rho\mu/2) , \tag{33}$$

$$t_0 = 1 , \quad t_1 = -\ln r , \quad t_2 = 0 . \tag{34}$$

It is seen that, up to the present order of approximation, the effect of natural convection does not appear in the temperature field, and that the above obtained solutions agree with those for the forced convection problem.

Substitution of (29) and (33) into (27) gives

$$\nabla_\rho^2 \Psi_1 = -\frac{1}{2}\frac{\mathrm{Gr}}{\mathrm{Re}} K_1(\rho/2)\exp(\rho\mu/2)\sin\theta . \tag{35}$$

The solution of (35) satisfying the boundary condition at infinity is

$$\Psi_1 = \sum_{n=1}^{\infty} C_{on}\rho^{-n}\sin n\theta - \frac{\mathrm{Gr}}{\mathrm{Re}}\sum_{n=1}^{\infty}(-1)^n \frac{\rho}{2n}\left\{2K_1\left(\frac{\rho}{2}\right)I_n\left(\frac{\rho}{2}\right) + K_0\left(\frac{\rho}{2}\right)\left[I_{n-1}\left(\frac{\rho}{2}\right) + I_{n+1}\left(\frac{\rho}{2}\right)\right]\right\}\sin n\theta , \tag{36}$$

where C_{0n} are integral constants and $I_n(z)$ the modified Bessel function of the first kind. The matching condition (24) determines the integral constants as

$$C_{0n} = 0 \quad \text{for all } n , \tag{37a}$$

$$b_{01} = 1 + \frac{1}{2}\frac{\mathrm{Gr}}{\mathrm{Re}} , \quad b_{11} = \frac{1}{4}\frac{\mathrm{Gr}}{\mathrm{Re}} . \tag{37b}$$

Thus, ψ_0, ψ_1 and Ψ_1 have been determined completely as

$$\psi_0 = \left(1 + \frac{1}{2}\frac{\mathrm{Gr}}{\mathrm{Re}}\right)\left(r - \frac{1}{r}\right)\sin\theta , \tag{38}$$

$$\psi_1 = \frac{\mathrm{Gr}}{\mathrm{Re}}\left(-\frac{1}{2} r\ln r + \frac{1}{4}\left(r - \frac{1}{r}\right)\right\}\sin\theta , \tag{39}$$

$$\Psi_1 = -\frac{\mathrm{Gr}}{\mathrm{Re}}\sum_{n=1}^{\infty}(-1)^n\frac{\rho}{2n}\left[2K_1\left(\frac{\rho}{2}\right)I_n\left(\frac{\rho}{2}\right) + K_0\left(\frac{\rho}{2}\right)\left\{I_{n-1}\left(\frac{\rho}{2}\right) + I_{n+1}\left(\frac{\rho}{2}\right)\right\}\right]\sin n\theta . \tag{40}$$

When $\mathrm{Gr}/\mathrm{Re} = 0$, these solutions become identical with the well-known potential flow solution. It is seen from (38) that, for $\mathrm{Gr}/\mathrm{Re} < -2$, the sign of the values of ψ_0 become negative, that is, the direction of the first-order inner flow is opposite to that of the main flow.

From the solutions obtained so far, the expressions for the radial and tangential velocity components may be calculated as

$$u = -\left(1 + \frac{\mathrm{Gr}}{2\mathrm{Re}}\right)\left(1 - \frac{1}{r^2}\right)\cos\theta - \Delta\frac{\mathrm{Gr}}{\mathrm{Re}}\left\{-\frac{1}{2}\ln r + \frac{1}{4}\left(1 - \frac{1}{r^2}\right)\right\}\cos\theta + \cdots , \tag{41a}$$

$$v = \left(1 + \frac{\mathrm{Gr}}{2\mathrm{Re}}\right)\left(1 + \frac{1}{r^2}\right)\sin\theta + \Delta\frac{\mathrm{Gr}}{\mathrm{Re}}\left\{-\frac{1}{2}\ln r - \frac{1}{4}\left(1 - \frac{1}{r^2}\right)\right\}\sin\theta + \cdots , \tag{41b}$$

in the inner region, and as

$$u = -\cos\theta + \Delta\frac{\mathrm{Gr}}{\mathrm{Re}}\left[\frac{1}{2}K_1\left(\frac{\rho}{2}\right)\left\{\exp\left(\frac{\rho\mu}{2}\right) - I_0\left(\frac{\rho}{2}\right)\right\} + K_0\left(\frac{\rho}{2}\right)\sum_{n=1}^{\infty}(-1)^n\left\{I_{n-1}\left(\frac{\rho}{2}\right) - \frac{2n}{\rho}I_n\left(\frac{\rho}{2}\right)\right\}\cos n\theta\right] + \cdots , \tag{42a}$$

$$v = \sin\theta + \Delta\frac{\mathrm{Gr}}{2\mathrm{Re}}K_0\left(\frac{\rho}{2}\right)\exp\left(\frac{\rho\mu}{2}\right)\sin\theta + \cdots , \tag{42b}$$

in the outer region.

4. Discussions

We shall first show some results for the velocity distribution calculated from (41) and (42). The distribution of u at $\theta = 0$ and v at $\theta = \pi/2$ is shown for $Pe = 0.1$ with Gr/Re as a parameter in Fig. 1 and that for $Gr/Re = 3$, -1 and -3 with Pe as a parameter in Figs 2(a) and 2(b). The velocity distribution shown in these figures is calculated from composite solutions, which can be obtained by adding the inner and outer solutions and then subtracting the common part. It is seen from Fig. 1(b) that the tangential velocity decreases with decrease in Gr/Re and becomes negative near the surface when the value of Gr/Re

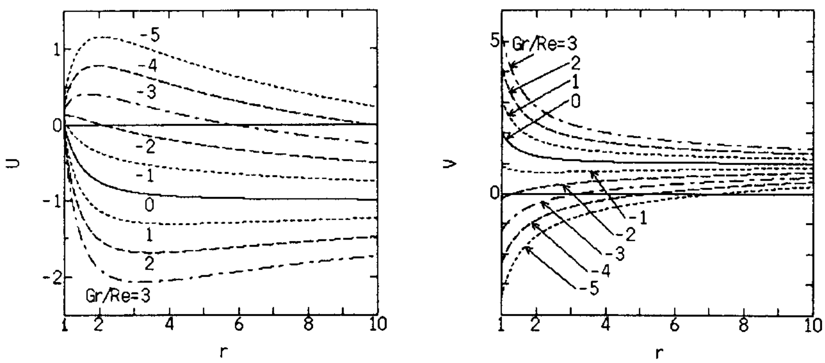


Fig. 1. Velocity distribution for $Pe = 0.1$, (a) normal velocity at $\theta = 0$, (b) tangential velocity at $\theta = \pi/2$.

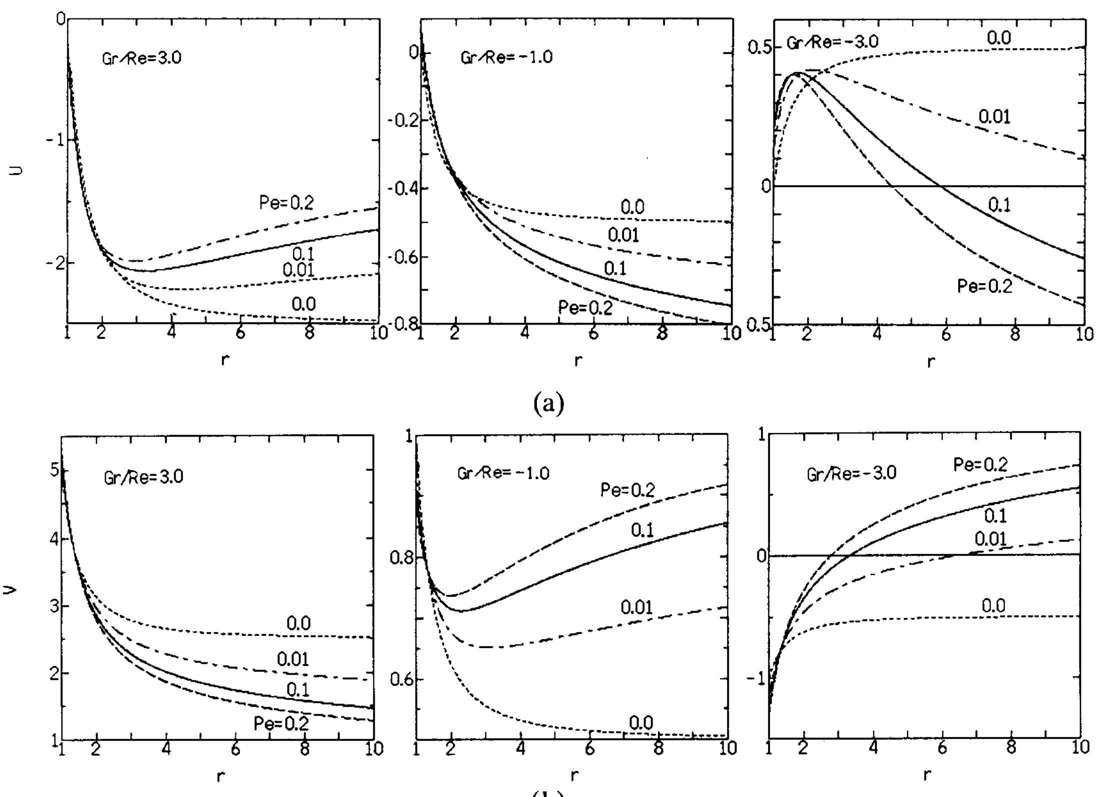


Fig. 2. Velocity distribution for $Gr/Re = 3$, -1 and -3, (a) normal velocity at $\theta = 0$, (b) tangential velocity at $\theta = \pi/2$.

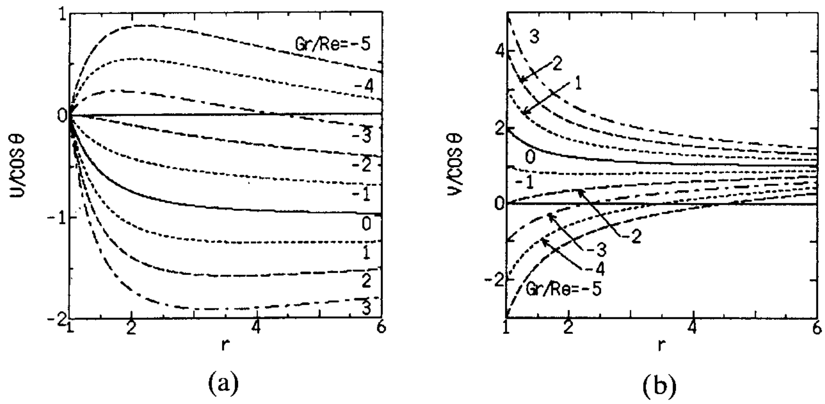


Fig. 3. Inner velocity distribution for Pe = 0.1, (a) $u/\cos\theta$, (b) $v/\sin\theta$.

becomes smaller than some negative value. This suggests that, for Gr/Re smaller than this critical value, vortices are formed at both sides of the cylinder. We should note at this point that the normal velocity shown in these figures does not satisfy the boundary condition on the surface, namely, $u = 0$ at $r = 1$. This is because the common part of the first-order outer solution which is of $O(\Delta^2)$ in magnitude in the inner region has not yet been subtracted from the composite expansion. Therefore, in order to see the velocity distribution near the surface, it is better to use the inner solution than to use the composite solution. Figure 3 show the distribution of $u/\cos\theta$ and $v/\sin\theta$ for Pe = 0.1 calculated from the inner solution (41). It is seen that the critical value of Gr/Re stated above is exactly −2. We can easily see from (41) that, up to the second order of approximation, this critical value is independent of Pe. This fact can be seen from Figs 4(a) and 4(b) also, in which inner velocity distribution for Gr/Re = 3, −1, −2 and −3 is shown with Pe as a parameter. Furthermore, we can see in Fig. 4 that the tangential velocity on the surface does not depend on the value of Pe up to the second order of approximation.

Next, we shall show some examples of the streamline pattern. Figs 5(a)–5(e) show the streamline patterns at Pe = 0.1 and for various values of Gr/Re calculated from the inner solution. It is seen that, for Gr/Re < −2, there exist two vortices at both sides of the cylinder which are surrounded by main flow, and that the streamline separating the vortices and the main flow is a circle, the radius of which becomes larger as the value of |Gr/Re| increases. Figure 6 shows streamline patterns at Gr/Re = −2.3 and Pe = 0.05. From this figure together with Fig. 5(d), we can see that the radius of the streamline separating the vortices and the main flow is larger for smaller values of Pe. In order to see the streamline patterns far from the cylinder, the streamlines calculated using the outer solution are shown at Pe = 0.1 and for Gr/Re = 3 and −2 in Fig. 7. It is seen that, for parallel flow (Gr/Re > 0), the fluid particles coming from the lower half space are drawn towards the cylinder, while that, for contra flow (Gr/Re < 0), the particles are displaced outwards. These facts are because the buoyancy force accelerates the fluid near the surface for parallel flow and decelerates it for contra flow. Moreover, we can see in these figures that, for parallel flow, the velocity of the upper stream is larger than that of the downstream, and that the converse holds for contra flow.

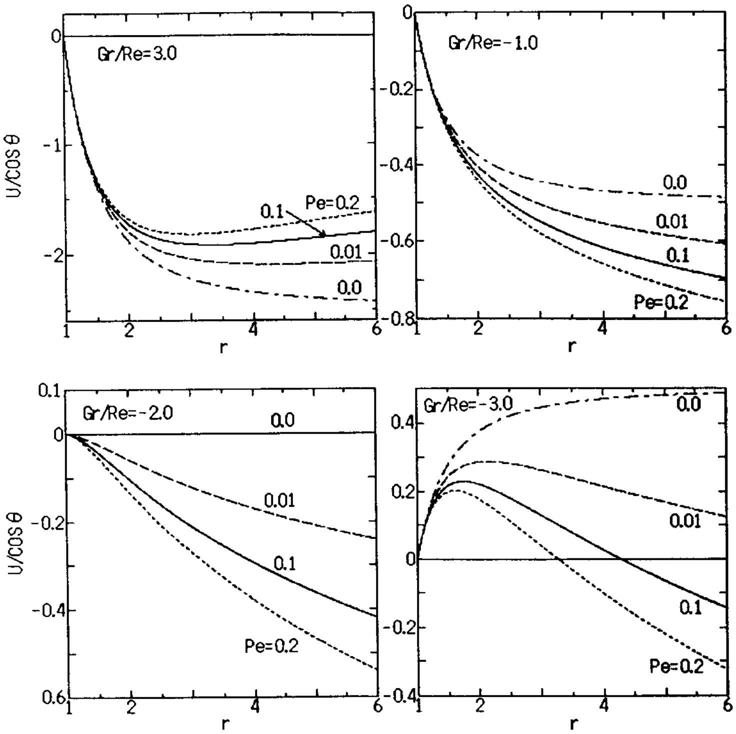


Fig. 4(a).

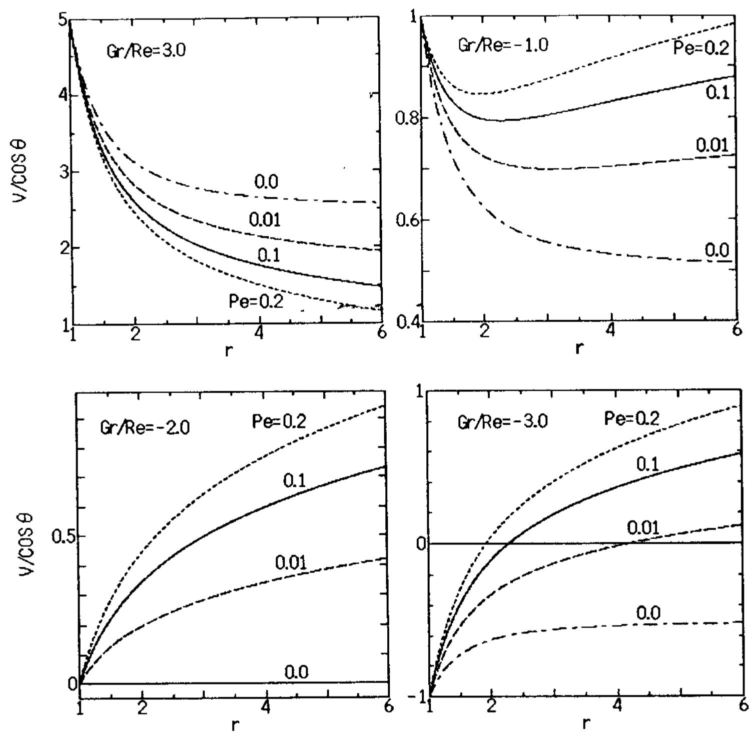


Fig. 4(b).

Fig. 4. Inner velocity distribution for Gr/Re = 3, −1, −2 and −3, (a) $u/\cos\theta$, (b) $v/\sin\theta$.

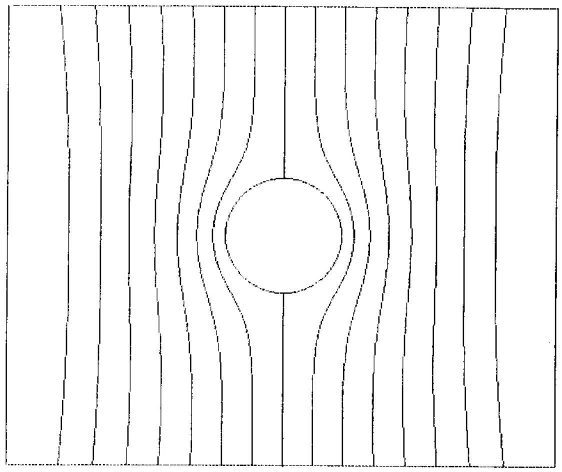


Fig. 5(a).

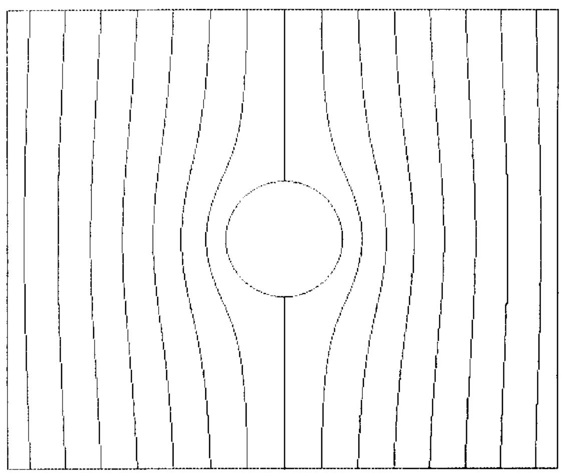


Fig. 5(b).

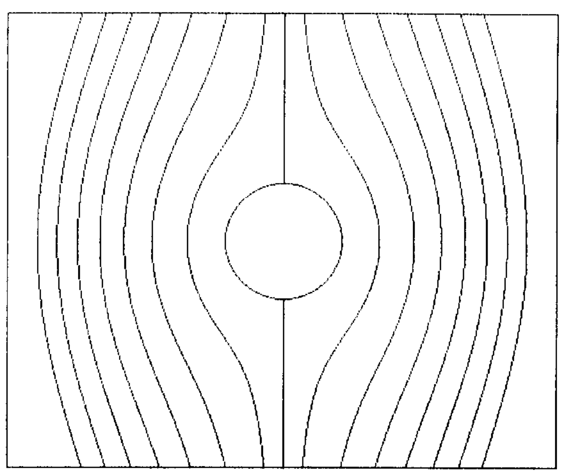


Fig. 5(c).

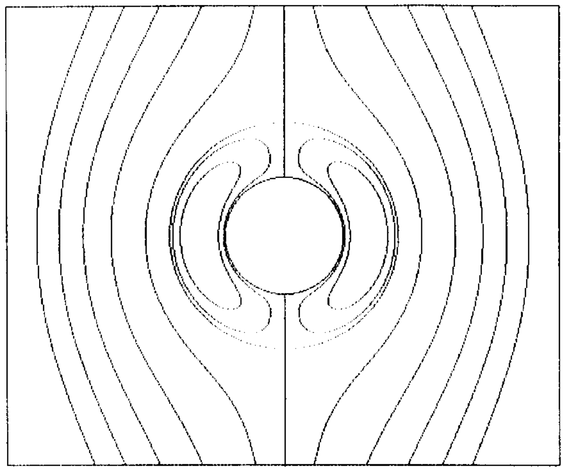


Fig. 5(d).

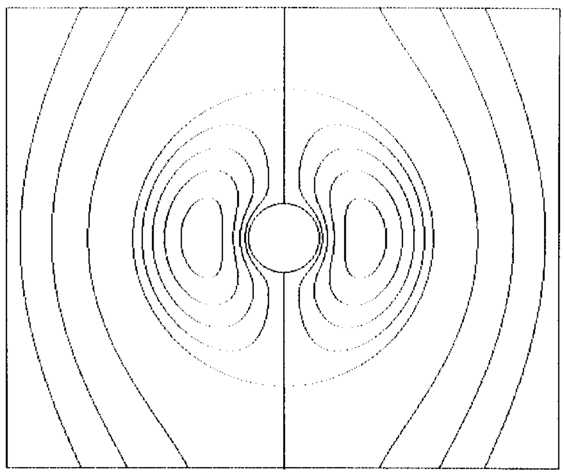


Fig. 5(e).

Fig. 5. Inner streamline pattern for Pe = 0.1, (a) Gr/Re = 3, (b) Gr/Re = 0, (c) Gr/Re = −2, (d) Gr/Re = −2.3, (e) Gr/Re = −3.

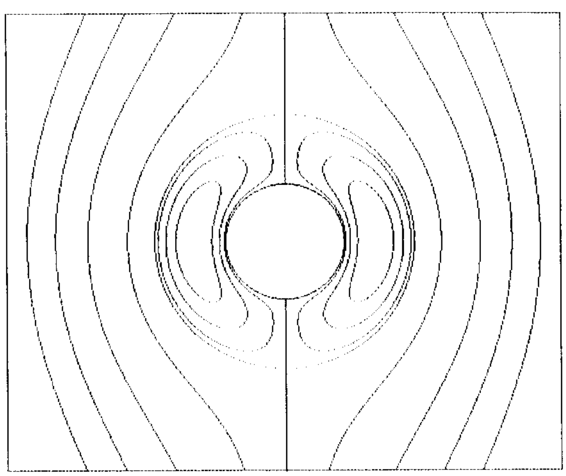
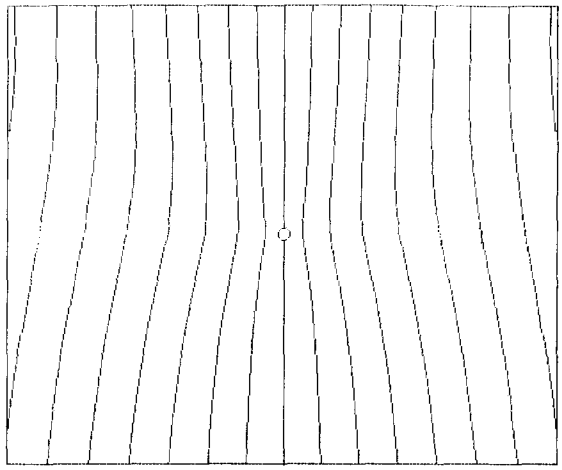


Fig. 6. Inner streamline pattern for Pe = 0.05 and Gr/Re = −2.3.

(a)



(b)

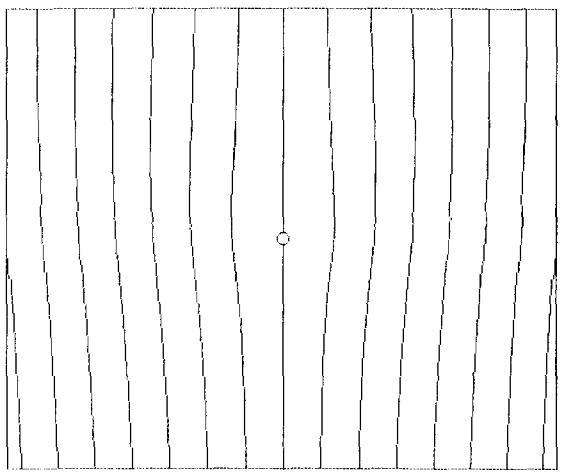


Fig. 7. Outer streamline pattern for Pe = 0.1, (a) Gr/Re = 3, (b) Gr/Re = −1.

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